CREDIT and BUBBLES

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Abstract

This paper is based on a production economy, where the capital of firms is composed of equity and debt. Firms face stochastic investment opportunities, and their debts are secured by the market price of the equity of the firms. We show in this paper how credit constraints and investment opportunities generate different prices of firms. One pricing corresponds to the case where the Tobin’s Q is one, and the market value corresponds to the book value. In the other case, the Tobin’s Q is larger, and the market value is larger. This allows firms with low amounts of capital to relax credit constraints. We show that price of firms may be bubbly, though this situation is rare. Finally, we test the effect of interest rate shocks on the value of firms, especially on their ability to generate crashes on market prices.

1 Related Works

As explained by Blanchard and Watson (1982), the price of an asset should reflect market fundamentals under rational behaviors and rational expectations of agents. Precisely, the price of an asset should depend exclusively on the information available about its current and future returns. If the price differs from its fundamental value, it means that there exists a rational deviation that we can name “bubble”. Blanchard and Watson (1982) deal with the existence of rational bubbles because irrational ones might be even more difficult to modelize, even if historical large bubbles seem to carry an undeniable part of irrationality. We choose to adopt this quite standard definition of bubble: when the price of an asset exceeds the theoretical sum of the stream of dividends also called fundamental price, the remaining part is called a bubble. In the following of our analysis, an asset may also represent the equity of a firm. In this case, cash flows replace dividends.

Many authors have been looking for bubbles in usual economic and financial models: either models with infinitely-lived agents like Tirole (1982), or models with overlapping generations like Tirole (1985). Especially in those models, as soon as money has a net positive price, it can be considered as a bubble because it never delivers any dividend. Santos and Woodford (1997) formalize the general case and point out the difficulty of ruling out bubbles when the time is infinite. The well known transversality condition is also
a formidable protection against bubbles, as proven for example by Obstfeld and Rogoff (1983).

The framework we use in our research was initially explicitly introduced by Kiyotaki and Moore (1997). Their article deals with credit, borrowing limits, and their effects on the economy, especially on prices. They study how credit may affect the economy through a model of lenders-borrowers in a production economy. The credit is subject to a collateral constraint: the land, which is also a production factor. The price of the collateral play therefore a double role, representing both the production factor and the limit of the loan. They also introduce the new idea that investment may not be automatic at each period but limited to a fraction of the producers, and this phenomenon changes the length of their cycles. These two hypotheses are the basis of a new generation of models on bubbles.

Considering asset prices not only as values but also as credit collateral became a main hypothesis. The credit is useful on the asset that is also collateral, this self-acting phenomenon may help distortions emerge in prices, among which bubbles. For example, subprime loans were mortgage loans, that does not help to stabilize prices when demand is crashing. Among the recent papers, Kunieda (2008) extended Tirole (1985) by adding borrowing constraints and analysed how bubbles on money can be inefficient, and what governments may do to correct the inefficiency. Kocherlakota (2009) developed a model where limiting the debt to the value of land generates a bubble on the prices. He also studied the consequences of prices crashes on the whole economy.

Very recently, Miao and Wang (2011) consider a production economy where the values of firm themselves are collateral for borrowing. To model restrictions on investment, they introduce, as Kiyotaki and Moore (1997), a constant probability of investment that each firm faces at each period. They allow firms to take one period loans to realize bigger investments, when the opportunity happens. These loans are guaranteed by the values of the firms themselves, and limited to a fraction of the value of the firms.
When the investment probability is sufficiently low, firms are willing to make bigger investments when the can. When the debt limitation is tight – which limits the investment amount – firms have to be “overpriced” to achieve their optimal investments. Actually, they show that firms can be priced two different ways, one is proportional to the capital: \( V(K) = vK \), and the other one has a net positive “shift”: \( V'(K) = v'K + b \). They call the \( b \) a “bubble”. Though quite efficient, this model appears to have little weaknesses: weak numerical plausibility and unusual form of the debt constraint. The objective of this paper is to widen this work and check the existence of bubbles in a different framework, where there exists a debt market.

We choose to adopt this approach mixing a production economy, possible restrictions in the investment, credit constraints. We impose a different structure of capital of the firms and calculate the equilibrium prices of firms. The results are a little different: there is a unique pricing, without any shift. The strategy of maximal investment is possible even for high values of the probability of investment, the different collateral constraints lead to the same pricing, and overall the prices of firms depend widely on interest rates.

Instead of solving an intertemporal consumption problem for an infinitively lived agent,

We adopt the investors’ point of view, which act to maximize the values of the firms. Therefore we solve the “firm’s problem”, but this is still a part of a general equilibrium. We adopt a discrete time formulation rather than a continuous time model to get better intuitions and results in the numerical part.

2 Model

2.1 General informations

We adopt a standard framework, also used by Miao and Wang (2011). We assume that there exist a continuum of firms on \([0, 1]\) with the same production function “Cobb Douglas”. \( Y_t^m \) is the production of firm \( m \), \( K_t^m \) and
\( N^m_t \) are respectively the capital and labor of the firm \( m \).

\[
Y^m_t = (K^m_t)^\alpha (N^m_t)^{1-\alpha},
\]

with \( \alpha \in ]0, 1[ \).

We suppose that the labor market is competitive, as a consequence, there is a common wage \( w_t \) per unit of labor. We also suppose that this wage is fixed exogenously. We can derive it from solving the usual intertemporal consumption problem. We also suppose that the level of labor perfectly adjusts to the demand of the firms.

Given the wage \( w_t \) and the competitive labor market, each firm maximizes at each period the cash flow of period \( t \) by adjusting its level of employment. The firm solves the following maximisation problem:

\[
\max_{N^m_t} (K^m_t)^\alpha (N^m_t)^{1-\alpha} - w_t N^m_t.
\]

This delivers the optimal level of employment for firm \( m \):

\[
N^m_t = \left( \frac{w_t}{1-\alpha} \right)^{-\frac{1}{\alpha}} K^m_t.
\] (1)

We can deduce the rate of return on the capital, which is the same for all firms:

\[
R^m_t = R_t = \alpha \left( \frac{w_t}{1-\alpha} \right)^{-\frac{1}{\alpha}}.
\] (2)

The next step is to study the dynamic of the capital of the firms. Usually the evolution of the capital of a firm is subject to the depreciation rate of the capital \( \delta \) and the periodic investment of the firm. This requires the ability of the firm to realize an investment at each period - each year in the present case -. The firms of the economy have different sizes. Precisely we know that the distribution of their sizes follows a Pareto law. There is a priori no reason why each firm could realize an investment at each period but on the contrary face difficulties like prohibitive costs or legal barriers. To model this irregular investment, we introduce a probability of investment \( \pi \) as done in Miao and Wang (2011) but also in Kiyotaki and Moore (1997) and Kocherlakota (2009). Such a probability should rely on the size of the firms: the biggest firms should be able to invest almost every
time whereas the smallest would have little possibilities of investment. For simplicity reasons, we consider this investment probability as a constant. Instead of just a probability of investment, Kunieda and Shibata (2012) use a random variable, but we do not require such sophistication to get results. We remember that the results of this essay will be valid only for economies including a large number of middle-sized firms. We will show in the following how this investment probability is a sine qua non condition for the existence of bubbles.

Suppose that each firm has an opportunity to make an investment at each period with a probability $\pi$. If $\pi = 0$, there is no investment possibility, if $\pi = 1$, the firm can invest at each period. When the investment is possible, firm $m$ chooses the amount of investment $I_m^t$. We deduce the equation of variation of the capital:

- $K_{m+1}^t = (1 - \delta) K_m^t$ with a probability $(1 - \pi)$;
- $K_{m+1}^t = (1 - \delta) K_m^t + I_m^t$ with a probability $\pi$;

where $\delta$ represents the depreciation rate of the capital.

We aggregate the firms to get the evolution of the global capital. By equation (1) we know that the capital labor ratio is identical for each firm. By aggregating we deduce that:

$$K_t = N_t \left( \frac{w_t}{1 - \alpha} \right)^{\frac{1}{\alpha}}.$$

We can replace this in equation (2) and we obtain the aggregate version:

$$R_t = \alpha K_t^{\alpha - 1} N_t^{1 - \alpha}. \quad (3)$$

We assume that the households supply labor inelastically: $N_t = \int N_t^m dm = 1$ and get $R_t = \alpha K_t^{\alpha - 1}$. In spite of the fact that the labor market is supposed to be perfectly competitive, we require this strong hypothesis to get results on the global capital.
The aggregate output $Y_t$ can be also simplified:

$$Y_t = \int (K_t^m)^\alpha (N_t^m)^{1-\alpha} dm = \int \left( \frac{K_t^m}{N_t^m} \right)^\alpha N_t^m dm = \left( \frac{K_t}{N_t} \right)^\alpha \int N_t^m dm = K_t^\alpha N_t^{1-\alpha}.$$ 

Each firm is owned by one or many risk neutral investors and has a quotation at time $t$: $V_t(E_t^m)$ on the stock market that logically represents the discounted sum of the future cash flows. We suppose that the investors are risk neutral, because there is no alea on the production functions.

The novelty of this paper relies on the the new structure of the capital of the firms, which is henceforth composed of equity and debt. For a firm $m$, the equity is denoted by $E_t^m$ and the debt by $D_t^m$. We have $K_t^m = E_t^m + D_t^m$. The debt is supplied perfectly elastically by the bank, which also fixes the interest rate at each period. The firms can only borrow from the bank, they have no access to any kind of financial investment.

### 2.2 Choice of the borrowing constraint

There must be a warranty for the bank when the firm borrows, otherwise the firm would never refund the bank. The firm logically pledges a part of the wealth. The physical capital $K_t^m$ is “physically” pledgeable, but the stock $V_t(K_t^m)$ is much better because it is priced and its value can be compared to the amount of the loan. The bank may not allow the loan to cover the whole capitalization of the firm but a portion of it, for economic, risk or legal reasons: there exists a parameter $\gamma$ that limits the borrowing.

Two natural constraints are possible, the bank may limit the loan either by $\gamma V_t(K_t^m)$ or by $V_t(\gamma K_t^m)$. These two constraints have been used in the recent litterature. The first one $L_t \leq \gamma V_t(K_t^m)$ has been used by Kiyotaki and Moore (1997). It means that the loan is guaranteed by the liquidation value of a part of the stock value of the firm. The second one $L_t \leq V_t(\gamma K_t^m)$ has been used by Miao and Wang (2011) and they explain that the loan is guaranteed by the value of a small and reorganized firm whose real capital would be $\gamma K_t^m$. These two approaches could somehow correspond to different countries where laws protect more either workers or investors. As we shall see, they lead to different results on the way of pricing the capital of firms. This is quite difficult to make a choice about the form of the constraint. Shleifer
and Vishny (1992) studied the liquidation values of assets, who appear to be variable and especially depending on the global context of the economy. At first sight it seems that the constraint introduced by Miao and Wang (2011) is a little less realistic that the other one. However the form of their constraint is the key evolution that allow them to find what they call bubbles. Whatever the constraint’s choice, it requires the uniqueness of the cotation at each period for all firms and for any level of capital. Indeed the cotation of two firms with the same level of capital is the same because we already assumed that all the firms have the same production function. However the quotation is not supposed to be linear, so $\gamma V_t(K^m_m) \neq V_t(\gamma K^m_m)$.

The bank only delivers loans and is neither supposed to be able to trade any form of stocks nor able to own a firm. If the firm can not refund the loan at the following period, it does not necessarily mean that the investors - owners of that firm are willing to sell a part of their stocks $\gamma V_t(K^m_m)$, they could also choose to change the structure of their firm by selling a part of physical capital $\gamma K^m_m$ which cotation on the market would be $V_t(\gamma K^m_m)$.

We test the two constraints on the model. It is interesting to check wether they deliver similar results and if the existence of a bubble relies on the constraint choice,

In our model, the collateral constraint obviously concerns the equity $E^m_m$. All along this chapter we consider the two “à la mode” constraints that are used in the recent litterature.

- the KM (Kiyotaki and Moore (1997)) constraint:
  $$D^m_t \leq \gamma V_t(E^m_t),$$

- the MW (Miao and Wang (2011)) constraint:
  $$D^m_t \leq V_t(\gamma E^m_t).$$

We suppose that the rental rate of capital is higher than the bank interest rate, otherwise the firm would never borrow from the bank: $R_t > r_t$. The financial cash-flow at each period is the yield on whole capital minus the interests of the debt:

$$R_t K^m_t - r_t D^m_t = R_t (E^m_t + D^m_t) - r_t D^m_t,$$

$$= R_t E^m_t + (R_t - r_t) D^m_t.$$
While \( R_t - r_t > 0 \) capital gains on the debt part are strictly positive, because the objective function equation (5) is linear in \( D^m_t \), and \( R_t \) does not depend on firm \( m \):

\[
\max_{D^m_t} R_tK^m_t - r_tD^m_t. \tag{6}
\]

As soon as \( R_t - r_t > 0 \), the firm borrows as much as possible. This remains true, whatever the choice of the constraint. When \( R_t - r_t > 0 \) the collateral constraint (MW or KM) is binding.

At each period the whole capital depreciates, \( K^m_t \) becomes \( K^m_t(1 - \delta) \). The capital is composed of equity and debt, therefore the sum \( E^m_t + D^m_t \) depreciates. As we just proved, at each time, the borrowing constraint must be binding. For the sequel of the chapter, we adopt KM constraint: at any time \( t \) the debt of the firm must satisfy \( D^m_t = \gamma V^m_t(E^m_t) \). At time \( t + 1 \) the capital of the firm satisfies the same relation: \( D^m_{t+1} = \gamma V^m_{t+1}(E^m_{t+1}) \).

The values of \( E_{t+1} \) and \( D^m_{t+1} \) depend on the possibility of investment. In addition, if there is a variation of the debt: \( D^m_{t+1} \neq D^m_t \), there must be a transfer to the bank. This variation \( (D^m_{t+1} - D^m_t) \) can be included either in the cash flow of the period, or from the next period capital.

Normally, when a firm wants to refund a bank, it uses the capital gains. If the variation of the debt \( (D^m_{t+1} - D^m_t) \) is included in the cash flow, we get the following equation on the investment:

\[
0 \leq I^m_t \leq R_tK^m_t - r_tD^m_t + D^m_{t+1} - D^m_t \leq R_t(E^m_t + \gamma V^m_t(E^m_t)) - r_t\gamma V^m_t(E^m_t) + \gamma V^m_{t+1}(E^m_{t+1}) - \gamma V^m_t(E^m_t). \tag{7}
\]

The investment \( I^m_t \) is the control variable of the firm. To solve the problem, the control variable must not depend on the value of next period state variable. However, equation (7) shows that \( I^m_{t+1} \) is correlated to \( E^m_{t+1} \) by term \( \gamma V^m_{t+1}(E^m_{t+1}) \). In addition \( \frac{\partial}{\partial E_{t+1}} V_{t+1}(E^m_{t+1}) > 0 \), because increasing \( I^m_{t+1} \) increases \( E_{t+1} \), which in turns increases the maximal bound of the investment \( I^m_{t+1} \). This allows for a Ponzi scheme. Technically, this means that the correspondence is not compact.

Because of this technical point, we must include the variation of the debt in the next period capital \( K^m_{t+1} \). The value of the debt at time \( t + 1 \) is different wether the investment happens \( D_i(t+1) \) or not \( D_n(t+1) \):
\begin{itemize}
  \item $K_{t+1}^m = (1 - \delta)K_t^m + D_{n(t+1)}^m - D_t^m$ with a probability $(1 - \pi)$, \hspace{1cm}\text{(8)}
  \item $K_{t+1}^n = (1 - \delta)K_t^n + P_t^n + D_{i(t+1)}^n - D_t^m$ with a probability $\pi$. \hspace{1cm}\text{(9)}
\end{itemize}

These two equations are analogous to (??) and (??) of the previous chapter.

The investment is bounded from above by the financial cash-flow of the period:

$$0 \leq I_t^m \leq R_tE_t^m - r_tD_t^n = R_tE_t^m + (R_t - r_t)D_t^n. \hspace{1cm}\text{(10)}$$

We can write the Bellman equation. We are pricing the equity $E_t^m$ instead of the whole capital.

$$V_t(E_t^m) = \max_{I_t^m \text{ satisfying eq. (10)}} R_tE_t^m + (R_t - r_t)D_t^n - \pi I_t^m$$

$$+ \pi \beta V_{t+1}(E_{i(t+1)}^m)$$

$$+ (1 - \pi) \beta V_{t+1}(E_{n(t+1)}^m); \hspace{1cm}\text{(11)}$$

where $E_{i(t+1)}^m$ and $E_{n(t+1)}^m$ are the respective values of the equity whether the investment occurs or not. $R_tE_t^m + (R_t - r_t)D_t^n - \pi I_t^m$ represents the cash-flow of period $t + 1$ minus the investment. We determine the next-period values of the equity:

$$E_{n(t+1)}^m + D_{n(t+1)}^m = K_{n(t+1)}^m = (1 - \delta)K_t^m + D_{n(t+1)}^m - D_t^m. \hspace{1cm}\text{(12)}$$

This gives:

$$E_{n(t+1)}^m = E_t^m (1 - \delta) - \delta D_t^m. \hspace{1cm}\text{(13)}$$

The same way, we get the equation on $E_{i(t+1)}^m$:

$$E_{i(t+1)}^m = E_t^m (1 - \delta) - \delta D_t^m + I_t^m. \hspace{1cm}\text{(14)}$$

This leads to the complete Bellman equation of the price of the equity:
\[ V_t(E_t^m) = \max_{0 \leq I_t^m \leq R_t E_t^m + (R_t - r_t) D_t^m} \left( R_t E_t^m + (R_t - r_t) D_t^m - \pi I_t^m \right. + \pi \beta V_{t+1}(E_{t+1}^m(1-\delta) - \delta D_{t+1}^m + I_{t+1}^m) + (1 - \pi) \beta V_{t+1}(E_{t+1}^m(1-\delta) - \delta D_{t+1}^m) \right. \]

The transversality condition is:
\[ \beta^t V_t(E_t^m) \xrightarrow{t \to \infty} 0. \]  

To solve this equation, we distinguish two main cases, when the solution is interior, and when the investment constraint is binding.

3 Solving the Bellman equation

3.1 Interior solution

Interior solution means that the constraint of the control variable will never be binding: \( I_{t+1} < R_t E_t^m + (R_t - r_t) D_t^m \) and the first order conditions are satisfied. Even if it does not create any bubble, it remains very useful to get an intuition on the form of the value function \( V_t(E_t^m) \). We adopt the Kiyotaki and Moore constraint: \( D_t^m \leq \gamma V_t(E_t^m) \). As explained before this is an equality \( D_t^m = \gamma V_t(E_t^m) \) because the borrowing constraint is binding (6). The MW case gives the same results. The Bellman equation becomes:

\[ V_t(E_t^m) = \max_{0 \leq I_t^m \leq R_t E_t^m + (R_t - r_t) \gamma V_t(E_t^m)} \left( R_t E_t^m + (R_t - r_t) \gamma V_t(E_t^m) - \pi I_t^m \right. + \pi \beta V_{t+1}(E_{t+1}^m(1-\delta) - \delta \gamma V_t(E_{t+1}^m) + I_{t+1}^m) + (1 - \pi) \beta V_{t+1}(E_{t+1}^m(1-\delta) - \delta \gamma V_t(E_{t+1}^m)) \right). \]

The first order condition with respect to the investment:
\[ V_t'(E_t^m) = \gamma V_t(E_t^m) + I_t^m \]

where \( V'(x) \) is the first order derivative of \( V \). The FOC with respect to the
state variable $E_t^m$ gives:

$$
V_t'(E_t^m) = R_t + (R_t - r_t) \gamma V_t'(E_t^m) \\
+ \pi \beta V_{t+1}'(E_t^m (1 - \delta) - \delta \gamma V_t(E_t^m) + I_t^m) (1 - \delta - \delta \gamma V_t'(E_t^m)) \\
+ (1 - \pi) \beta V_{t+1}'(E_t^m (1 - \delta) - \delta \gamma V_t(E_t^m)) (1 - \delta - \delta \gamma V_t'(E_t^m)).
$$

(19)

Substitute (18) in (19) and we get:

$$
R_t + (R_t - r_t) \frac{\gamma}{\beta} = \frac{1 - \beta}{\beta} + \delta + \frac{\delta \gamma}{\beta}.
$$

(20)

We remark that the rental rate of the capital does not depend on $\pi$ but on the borrowing limit $\gamma$. Actually this is logical: the investment constraint is assumed not to be binding, which means that the firm can reach the optimal level of investment. The probability of investment does not affect the average level of capital, and the rental rate of the capital. We understand that $R_t$ has to cover the subjective interest rate of the investor: $\frac{1-\beta}{\beta}$ and the depreciation rate of capital $\delta$. The rental rate excess with respect to the debt $(R_t - r_t)$ associated to its proportion in the capital $\gamma$ has to cover the depreciation of the capital financed by the debt $\delta \gamma$. The rental rate of capital can be also expressed as:

$$
R_t = \frac{1 - \beta + \delta \beta + \delta \gamma + \gamma r_t}{\beta + \gamma}.
$$

(21)

For all values of $\gamma \in [0, 1]$, $R_t$ remains larger than $r_t$, the borrowing constraint keeps binding. When $r_t$ increases, $R_t$ also increases and the capital reduces, as well as the debt. Again, we need to be careful with long-term interpretations, depending on the validity of the discount condition $\beta = \frac{1}{1+r_c}$, where $r_c$ is the limit of the short-term bank interest rate $r_t$. The derivative of $R_t$ with respect to the borrowing parameter $\gamma$ is:

$$
\frac{\partial R_t}{\partial \gamma} = \frac{-1 + \beta + \beta r_t}{(\beta + \gamma)^2}.
$$

(22)

The positivity of the numerator depend on the link between $\beta$ and $r_t$: $\frac{\partial R_t}{\partial \gamma} > 0 \Leftrightarrow \beta > \frac{1}{1+r_t}$. When the subjective discount rate of the agents $r_c$ such that $\beta = \frac{1}{1+r_c}$ is lower than the bank interest rate $r_c < r_t$, investors reduce the
whole capital when the borrowing constraint is relaxed, they consume more. This is the opposite as the usual behavior of a household which stores a part of his wealth in a bank asset. Indeed, when the debt is more costly, the investors decrease the capital to pay less interests.

Increasing the bank interest rate reduces the capital of the firm, it reduces the investment, and the debt. Relaxing the debt limit (increasing $\gamma$) increases the level of capital if the bank interest rate is sufficiently low: $r_t < \frac{1 - \beta}{\beta}$.

We deduce from (18) that the valuation of the firms is given by:

$$V_t(E^m_t) = \frac{E^m_t}{\beta} + b_t; \quad (23)$$

where $b_t$ is a common constant that depends on the period. To find $b_t$ we substitute equation (23) in the Bellman equation (17):

$$b_t(1 - \gamma(R_t - r_t) + \delta \gamma) = \beta b_{t+1} \quad (24)$$

We use the equation (21) to get:

$$b_t \frac{1 + \gamma + \gamma r_t}{\beta + \gamma} = b_{t+1} \quad (25)$$

Because\(^1\) $1 + \gamma r_t > \beta$ the solution to equation (25) diverges. It can still be a solution to the Bellman equation, only if it satisfies the transversality condition, equation (16).

$$\beta^t V_t(E^m_t) = \beta^{t-1} E^m_t + \beta^t b_t. \quad (26)$$

The first term converges to zero, and the second is:

$$\beta^t b_t = \beta^t b_0 \prod_{t=0}^{t-1} \frac{1 + \gamma + \gamma r_t}{\beta + \gamma}$$

$$= b_0 \prod_{t=0}^{t-1} \frac{\beta + \beta \gamma + \beta \gamma r_t}{\beta + \gamma} \quad (27)$$

\(^1\)We assume $r_t > 0$.  

13
To know if equation (27) converges to zero, we must study the limit of the term \( \frac{\beta + \beta\gamma + \beta\gamma r}{\beta + \gamma} \). Let \( r_\infty \) be the limit of the short-term interest rate:

\[
\frac{\beta + \beta\gamma + \beta\gamma r}{\beta + \gamma} < 1 \iff \beta < \frac{1}{1 + r_\infty}.
\] (28)

Let \( r_c \) be the discount rate of the consumers: \( \beta = \frac{1}{1 + r_c} \). In this case, equation (27) converges to zero if and only if:

\[
r_c > r_\infty.
\] (29)

If equation (29) is true, the transversality condition (16) is true and any price of type:

\[
V_t(E^m_t) = \frac{E^m_t}{\beta} + b_0 \prod_{t=0}^{t-1} \frac{1 + \gamma + \gamma r_t}{\beta + \gamma}
\] (30)

is a solution to the Bellman equation (17). In addition this solution has a diverging bubble component in the price. To know if this solution exists, we must study the behavior of the bank.

In section 5.1.2, we proved that the firm was borrowing if and only if the rental rate of the capital \( R_t \) is larger than the bank interest rate \( r_t \). This condition when \( t \to \infty \) becomes:

\[
\frac{1 - \beta + \delta \beta + \delta \gamma + \gamma r_\infty}{\beta + \gamma} > r_\infty.
\] (31)

In this equation (31), we can substitute \( \beta = \frac{1}{1 + r_c} \). We get a new condition on \( r_c \):

\[
r_c + \delta + \delta \gamma > r_\infty.
\] (32)

which is compatible with the previous condition, equation (29). The remaining question is to determine the behavior of the bank.

### 3.2 The bank optimizes

To rule out bubbles, the bank can set the long-term interest rate to \( r_\infty > r_c \). However, if the bank optimizes the profits, the value of \( r_\infty \) is the solution to the bank’s problem:

The bank maximizes the limit of the sequence of one-period problems
by solving:

\[ \max_{r_\infty} r_\infty D_t, \quad (33) \]

with \( D_t \) such that \( K = E_t + D_t \) and \( R \) is given by equation (21): \( R = \frac{1-\beta+\delta\beta+\delta\gamma+\gamma r_\infty}{\beta+\gamma} \). To link \( E_t \) and \( D_t \), we consider equation (30). To simplify let us write \( B_t = B_0 \prod_{t=0}^{t-1} \frac{1+\gamma+\gamma r_t}{\beta+\gamma} \). Because \( R \) is constant, we know that \( K \) is constant. At each time \( t \) we have:

\[ D_t = \gamma V_t(E_t^m) = \gamma \frac{E_t^m}{\beta} + B_t, \quad (34) \]

Because \( V_t(E_t^m) \) diverges if \( E_t^m \geq 0 \), to keep \( D_t + E_t \) constant, we need to consider a negative equity:

\[ K = \gamma \left( \frac{E_t^m}{\beta} + B_t \right) + E_t \]

\[ \iff E_t = (K - \gamma B_t) \frac{\beta}{\beta + \gamma}. \quad (35) \]

The bank problem at time \( t \) becomes:

\[ \max_{r_\infty} r_\infty \gamma V_t \left( \left( K - \gamma B_t \right) \frac{\beta}{\beta + \gamma} \right) \]

\[ = \max_{r_\infty} r_\infty \frac{K}{\beta} + \frac{\beta B_t}{\beta + \gamma} \]

\[ = \max_{r_\infty} r_\infty \frac{1}{\beta} \frac{1}{\alpha} \frac{1 + \gamma + \gamma r_t}{\beta + \gamma} \prod_{t=0}^{t-1} \frac{1 + \gamma + \gamma r_t}{\beta + \gamma}. \quad (36) \]

The profit of the bank is strictly increasing in \( r_t \), which states that the bank should fix \( r_\infty \) as high as possible. However by equation (29), the firm borrows if and only if \( r_c > r_\infty \). There is no well defined solution to this problem, because the domain of \( r_\infty \) is open. Any value of \( r_\infty \) strictly lower but closed to \( r_c \) guarantees to the bank a large and increasing profit.

There is a state of the economy, such that the price of the equity includes a non-zero growing bubble. There is a Ponzi sheme on the debt due to the bubble, which still satisfies the transversality condition. This state generates a very unusual situation of negative equity for the firm. If we impose a reasonable condition of positive equity, this rules out the bubble in a finite time \( T \), when \( E_T^m = 0 \).

If we suppose that the firm has a long-term positive equity, the bubbly
term is zero: \( B_0 = 0 \). The price of the firm is therefore \( V_t(E_t) = \frac{E_t}{\beta} \). We can deduce the value of the debt: \( D_t = \gamma V_t(E_t) = \frac{\gamma E_t}{\beta} \). The value of the equity comes from the value of \( R \) in equation (21):

\[
K = \frac{R^{1-\alpha}}{\alpha} = D_t \left( 1 + \frac{\gamma}{\beta} \right)
\]  

(37)

We deduce that the debt is constant over time, and the value of the debt is:

\[
D = \frac{\beta}{\beta + \gamma} \left( \frac{1 - \beta + \delta \beta + \delta \gamma + \gamma r_\infty}{\alpha (\beta + \gamma)} \right)^{1/(\alpha - 1)}.
\]  

(38)

The optimization function of the bank is:

\[
\max_{r_\infty} \left[ r_\infty \frac{\beta}{\beta + \gamma} \left( \frac{1 - \beta + \delta \beta + \delta \gamma + \gamma r_\infty}{\alpha (\beta + \gamma)} \right)^{1/(\alpha - 1)} \right].
\]  

(39)

This gives the optimal long term interest rate of the bank, expressed with the discount rate of the investor \( r_c \):

\[
r_\infty = (r_c + \delta \beta + \delta \gamma) \frac{1 - \gamma}{1 - \gamma + \gamma \alpha}.
\]  

(40)

With the value of \( r_\infty \), we can deduce the value of \( R \), and the complete problem (investor, bank, consumer) has a unique solution.

- If the equity of the firms is positive, the KM constraint rules out the hypothetic constant term. As a consequence the valuation does not depend on the time, and becomes:

\[
V(E_i^{m}) = \frac{E_i^{m}}{\beta}.
\]  

(41)

The bank optimizes its profits by setting the long term interest rate. There is a unique solution to the model.

- If the firms could be in negative equity, there would exist a bubble in the prices of the firms. This bubble would be geometrically growing over time, and would create a Ponzi scheme on the debt. This would also allow for unlimited profits for the bank, while the long-term interest rate on debts would remain smaller than the subjective discount rate of the investor \( r_c \).
We proved in this part that the investment probability $\pi$ is “useless” when the investment constraint is not binding, i.e. when the shadow price of capital does not exceed $\frac{1}{\beta}$. We also remark the formulation of the value function in (23) that we will impose to find the boundary solution. If we adopt the Miao and Wang point of view: the constraint becomes $V_t(\gamma K_t^m)$, the results are the same because the derivative of $V_t(\gamma K_t^m)$ also gives a factor $\gamma$.

### 3.3 Maximal investment

In this part, we consider initially the Miao and Wang constraint\(^2\). We go back to the Bellman equation and now we suppose that the shadow price of the capital exceeds $\frac{1}{\beta}$: when the investment opportunity happens, the firm is willing to invest as much as possible, and the investment reaches its maximal level: $I_t^m = R_t E_t^m + (R_t - r_t)V_t(\gamma E_t^m)$, because $D_t^m = V_t(\gamma E_t^m)$.

We replace this expression in the Bellman equation and we also substitute the value function by $V_t(E_t^m) = v_t E_t^m + b_t$. We get the following equation:

$$v_t E_t^m + b_t = (R_t E_t^m + (R_t - r_t)(\gamma v_t E_t^m + b_t)) (1 - \pi) + \beta v_{t+1} (E_t^m(1 - \delta) - \delta (\gamma v_t E_t^m + b_t)) + \beta b_{t+1}$$

$$+ \pi \beta v_{t+1} (R_t E_t^m + (R_t - r_t)(\gamma v_t E_t^m + b_t)) \tag{42}$$

We identify the terms that depend on $E_t^m$ and the others to get two difference equations on $v_t$ and $b_t$:

$$v_t = \frac{\beta v_{t+1} (\pi R_t + 1 - \delta) + R_t (1 - \pi)}{1 - \gamma (R_t - r_t) (1 - \pi) + \beta \gamma v_{t+1} (\delta - \pi (R_t - r_t))}; \tag{43}$$

$$b_t (1 - ((R_t - r_t)(1 - \pi) + (R_t - r_t) \pi \beta v_{t+1} - \delta \beta v_{t+1})) = \beta b_{t+1}. \tag{44}$$

We still need to consider the equation on the global capital $K_t$ to get the complete dynamic system. We know that for each firm we have: $K_t^m = E_t^m + V_t(\gamma E_t^m)$ because the investment constraint is binding, so we need to

\(^2\)The KM case is done in Appendix C.
get the evolution of $E_t^m$:

$$E_{t+1}^m = (1 - \delta)E_t^m - \delta V_t(\gamma E_t^m) + \pi (R_t E_t^m + (R_t - r_t)V_t(\gamma E_t^m)). \tag{45}$$

We aggregate all the firms to get:

$$E_{t+1} = (1 - \delta)E_t - \delta(v_t \gamma E_t + b_t) + \pi (R_t E_t + (R_t - r_t)(v_t \gamma E_t + b_t)). \tag{46}$$

Since $K_t^m = E_t^m + V_t(\gamma E_t^m) = E_t^m(1 + \gamma v_t) + b_t$ we deduce the equation on the average capital:

$$\frac{K_{t+1} - b_{t+1}}{1 + \gamma v_{t+1}} = (1 - \delta) \frac{K_t - b_t}{1 + \gamma v_t} - \delta \left( v_t \gamma \frac{K_t - b_t}{1 + \gamma v_t} + b_t \right) + \pi \left( R_t \frac{K_t - b_t}{1 + \gamma v_t} + (R_t - r_t) \left( v_t \gamma \frac{K_t - b_t}{1 + \gamma v_t} + b_t \right) \right). \tag{47}$$

Instead of considering this “not so easy to use” equation, we focus on the equity equation (46) and make the link between $E_t$ and $R_t$:

$$R_t = \alpha K_t^{\alpha-1} = \alpha (E_t(1 + \gamma v_t) + b_t)^{\alpha-1}. \tag{48}$$

The whole system is now given by equations (43) and (44), (46) and (48). We impose the transversality conditions to the solutions.

$$\beta^t v_t E_t \underset{t \to \infty}{\rightarrow} 0, \quad \beta^t b_t \underset{t \to \infty}{\rightarrow} 0. \tag{49}$$

Given the complete system, we look for the equilibrium values of the variables. This allows to find the price(s), and the equilibrium global level(s) of capital of the production economy.

## 4 Linear pricing

### 4.1 Equilibrium values of the model

Taking $b = 0$ is a solution to equation (44). The solution of $b = 0$ is the same for both constraints (MW and KM) for linearity reasons. We use the
two equations (43) and (46) and get the following results. The variables without \( t \) subscript denote the steady-state values. We obtain the following system:

\[
v = \frac{R(1 - \pi)}{1 - \beta - \gamma(R - r_\infty)(1 - \pi)},
\]

\[
v = \frac{\delta - \pi R}{\gamma(R - r_\infty) - \delta}.
\]

Together they give the values of \( R \) and \( v \):

\[
R = \frac{\delta(1 - \beta) + \delta \gamma r_\infty(1 - \pi)}{\pi(1 - \beta)}, \quad (50)
\]

\[
v = \frac{\delta \gamma(1 - \pi)}{\gamma(\pi(1 - \beta) - \delta \gamma(1 - \pi))}, \quad (51)
\]

The rental rate of capital \( R \) is inversely proportional to the probability of investment \( \pi \). This shows that when investment opportunities rarefy, the rental rate of the capital increases, and the capital decreases. This corresponds to smaller firms with higher rate of return on the capital. When the collateral constraint \( \gamma \) is relaxed, the rental rate of the capital slightly increases. This corresponds to the interests on the depreciation of the debt \( \propto \delta r_\infty \).

Curiously the \( v \) shadow price of the capital does not depend on the bank interest rate. On Figure 1, we draw \( v \) as a function of \( \pi \) and \( \gamma \) to know when \( v > \frac{1}{\beta} \), necessary condition for maximal investment. On this graph and the following ones, we limit the values to 20, though it reaches higher values. The same way, the lower limit is 0 even if the values are negative. The dark blue area corresponds to the values of \( v \) such that \( v < \frac{1}{\beta} \), where the solution does not exist.

On Figure 1, we remark a curve along which \( v \) diverges. Looking at equation (51), we understand that when \( \pi \rightarrow \frac{\delta}{1 - \beta + \delta \gamma}, \quad v \rightarrow \infty \). We will consider how interpreting this phenomenon in the following. Above this curve, for any fixed value of \( \gamma \), \( v \) is decreasing with respect to \( \pi \) which means that there exists a threshold such that \( v = \frac{1}{\beta} \) and we reach the interior
solution. In this model, the combinations of \( \pi \) and \( \gamma \) that allow \( v > \frac{1}{\beta} \) do not especially correspond to restrictions in investment or collateral but on the opposite, the values are quite large. To check the validity of the borrowing constraint, we check that \( R \) remains larger than the long-term bank interest rate \( r_\infty = 4\% \) for any couple \((\pi, \gamma)\).

We can draw the whole domain where the steady state of the linear pricing exists.

In the blue zone, only the interior solution exists, \( v = \frac{1}{\beta} \). On the \( \pi \) upper limit of the red zone\(^3\), \( v = \frac{1}{\beta} \) and the value of \( R \) of the linear pricing correspond to the value of \( R \) of the interior solution. This means that when \( \pi \) increases, the equilibrium steady state of the linear pricing becomes *continuously* the steady state of the interior solution.

On the left border of the red zone, \( R \) behaves continuously while \( v \) diverges. This means that \( v\gamma \) diverges and \( K = E + D \) remains smooth. However \( E \geq 0 \) and \( D \geq 0 \). The equity of the firm \( A \to 0 \) and the shadow

\(^3\)Right border of the red part Figure 2.
Figure 2: Existence domain of the shadow price \( v \): \( v > \frac{1}{\beta} \) and \( R > r_\infty = 4\% \)

price \( v \) diverges to keep the product \( v\gamma A \) smooth. On this limit the equity converges to zero and the capital is only composed of debt. Along this curve, there exists a bubble. The value of the debt is \( v\gamma E = D \) and the value of the capital is \( K = E + D = 0 + D \). There is a bubble, and the price of the bubble is \( \frac{K}{\gamma} \).

The performance of this model relies on the ability of the economy and the prices to reach a steady state for quite large and realistic values of \( \pi \) and \( \gamma \), in which the capital is priced over its “usual” value: \( v > \frac{1}{\beta} \). There also exists a relation between the investment parameters: \( \pi = \frac{\delta \gamma}{1 - \beta + \delta \gamma} \) such that when it is satisfied, the capital of firms is only composed of debts, the amount of the equity is zero, but the equity has a net positive price, there is a real bubble in the economy. The value of the bubble is \( \frac{K}{\gamma} \).

By linearization and numerical tests, we prove that the steady state is stable where it exists.

4.2 The bank’s equilibrium interest rate

Again, to get a global equilibrium, let us consider that the bank optimizes its profits by setting the value of the debt interest rate. At each period, the global amount of the debt is \( v\gamma E \). The value of the capital is \( E + D = \)
$E(1 + v\gamma)$. We deduce $E$ from $R = \alpha K^{\alpha - 1} = \alpha E^{\alpha - 1}(1 + v\gamma)^{\alpha - 1}$. The bank problem can be stated this way:

$$\max_{r_\infty \text{such that eq.}(53)} r_\infty \gamma \frac{\delta \gamma (1 - \pi)}{\gamma (\pi (1 - \beta) - \delta \gamma (1 - \pi))} E.$$  \hspace{1cm} (52)

$$\frac{\delta (1 - \beta) + \delta \gamma r_\infty (1 - \pi)}{\pi (1 - \beta)} = \alpha E^{\alpha - 1}\left(1 + \gamma \frac{\delta \gamma (1 - \pi)}{\gamma (\pi (1 - \beta) - \delta \gamma (1 - \pi))}\right)^{\alpha - 1}$$ \hspace{1cm} (53)

When $E$ is fixed, increasing $r_\infty$ increases the bank’s profit. Looking at the constraint, an increase of $r_\infty$ leads to an increase of $E^{\alpha - 1}$. But $(\alpha - 1) < 0$, therefore increasing $r$ decreases $E$. To conclude there is a unique solution to the bank problem, which determines the level of capital of the firms.

We can study the whole equilibrium when the economy is located on the parameters’ bubble border: $\pi = \frac{\delta \gamma}{1 - \beta + \delta \gamma}$. We already know that $v = \infty$ and $E = 0$, but the product is bounded, as a consequence it satisfies the transversality condition. When replacing these equilibrium values of the variables in the Bellman equation (42), we deduce the the value of the equilibrium rental rate of the capital:

$$R - r_\infty = \frac{\delta}{\pi}. \hspace{1cm} (54)$$

By substitution with the analytical expression of $R$ equation (50), we can also determine the value of $R - r_\infty = \frac{\delta}{\pi} = \frac{1 - \beta + \delta \gamma}{\gamma}$. We deduce that the expression (54) is true whatever the value of the bank interest rate. The larger the amount of debt in the capital (\(\gamma\)), the larger the spread between the rental rate of the capital and the bank interest rate, and the smaller the amount of total capital. To maximize profits, the bank may also increase the long term interest rate $r_\infty$, but in turn this decreases the amount of capital, and especially the amount of debt.

On the following figure, we plot the value of the one-period banks-profit depending on the long term interest rate $r_\infty$. We take the following parameters values: $\alpha = 0.4$, $\gamma = 50\%$, $\delta = 3\%$ and we also suppose that the discount factor of the investor is related to the bank long term interest rate.
Figure 3: Profit of the bank depending on the long term interest rate, $\alpha = 0.4$, $\gamma = 50\%$, $\delta = 3\%$ by the classic condition $\beta = \frac{1}{1+\rho_{\infty}}$.

$$r_{\infty}K = r_{\infty}D = r_{\infty}\left(\frac{1}{\alpha}\left(\frac{1-\frac{1}{1+\rho_{\infty}}+\gamma\delta}{\gamma}+\rho_{\infty}\right)\right)^{\frac{1}{\alpha-1}} .$$ (55)

4.3 Interest rate shocks

In this section, we focus on the variations of prices and of capital when the capital is priced over $\frac{1}{\beta}$. This correspond to the part “linear pricing”.

We evaluate the impact of an unexpected shock on the bank’s interest rate on the price of the firm. We take $\alpha = 0.4$, $\beta = 0.96$, $\rho_{\infty} = 4\%$, $\gamma = 90\%$, $\pi = 40\%$. The interest rate mean is 4\% and is shocked by 0.5\% (absolute value). The steady state values are: $v = 6$, $R = 9.6\%$, $E = 1.68$. The price of the average equity is $V(E) = 10.07$. On the following Figures 4, 5 and 6 these steady-state values correspond to the red lines. We expose first the effect of a non-persistent shock, lasting 1 period, on Figure 4.

The unexpected shock on the interest rate decreases the equity, but increases the shadow price $v_t$. The average price of firms decreases by less than 0.1\%, because the effect on $v_t$ is slightly delayed compared to the one on the equity $E_t$. The rental rate of the capital logically increases, which means that the global capital first decreases, to refund the increase of the interests and then progressively goes back to its initial level.

The results are amplified if the shock is more persistent. Suppose that
Figure 4: Evolution of the equilibrium values in response to a shock of 0.5\% on the interest rate lasting 1 year.

The interest rate shock is decreasing (AR-1 process, with autoregression coefficient $\rho = 0.4$) lasting 5 years, on Figure 5. The responses of the shadow price $v_t$ and the equity $E_t$ are amplified, but on average, the price of the equity $V_t(E_t)$ is not changed compared to the shock lasting 1 period.

If the amplitude of the shock is increased, simulations show that the response of the equity and the price are also proportionally increased.

However, the price of the equity is highly impacted if the shock is lasting over more periods\(^4\). For example, on Figure 6, the shock is lasting 20 years ($\rho = 0.9$ in the AR-1 process).

The main difference, due to the persistence of the shock, is the initial

\(^{4}\text{We cannot study the effects of large interest and persistent interest rate shocks because we reach Dynare's limits.}\)
Figure 5: Evolution of the equilibrium values in response to a shock of 0.5% on the interest rate lasting 5 years and persistent decrease of the price $V_t(E_t)$ by more than 2%, which leads to think that there is a fall of 2% on the markets. Obviously, if the amplitude of the shock on the interest rate was higher, this would generate a larger fall.

Prices of firms are very responsive to interest rate shocks. The shadow price increases with a positive shock on the bank interest rate. It tends to counterbalance the decrease of the net equity. On average, market prices decrease by a small extent. The return to the equilibrium (prices and equity) is very long compared to the length of the shock. When the interest rate shock is more persistent, the shadow price first jumps down at the shock, like the equity. This can be considered as “stock market fall”.

4.4 Shocks on investment

We expose first the results of a 5% (absolute) shock on the limit of borrowing $\gamma$ lasting 5 years. We adopt the same parameters’ values as in the section 5.3.3 presenting shocks on the interest rate.

The effects are unpredictable. Indeed, if $\gamma$ increases, we are waiting for a decrease of $v_t$ to adjust the net borrowing $v_t\gamma E_t$. Actually the pricing $v_t$ increases and the equity reduces. This increase of 5% on $\gamma$ creates a 2% increase of $v_t$ and decreases $E_t$ by 0.4%. Though it might be difficult to understand the response of the economy, we remark that the price is more volatile than the equity. The rental rate of the capital does not change a lot, which means that the capital is almost constant. The equity reduces...
Figure 7: Evolution of the equilibrium values in response to a shock of 5% on the borrowing limit $\gamma$

and the debt increases for the shock. To conclude about the behavior of this linear pricing, we simulate the effect of a shock of 5% (relative shock of 11%) on the probability of investment $\pi$. 
The 5% shock on $\pi$ generates an increase of the equity. In addition, even if the shadow price decreases, there is an increase of the price of the equity, which means that the debt increases for the shock. The rental rate of the average capital decreases, which shows that the capital increases. A positive shock on the probability on investment means that it is easier to invest. As a consequence, the valuation of the firm can be relaxed a little bit, because the firm is able to invest more often. Even if the shadow price of the equity reduces, a positive shock on $\pi$ generates a global increase of market prices.

To conclude about the linear pricing, the volatility of the price and the equity is identical, both are highly impacted by interest rate variations. A positive shock on the interest rate reduces the equity, and reduces a little the
whole capital. The shadow price increases, except if the shock is persistent, in this case, it decreases first. Relaxing the borrowing limit increases the shadow price and reduces the average equity, and increasing the probability of investment increases the equity and decreases the shadow price. In all cases, the effect on the equity exceeds the effect on the shadow price, as a consequence, market prices evolve like the equity, but “smoother”.

5 Multiple equilibria

We are also interested by the existence of an affine pricing as solution to the maximal investment case, and we would like to see how it depends on the borrowing constraint.

The equilibrium values of the affine pricing exist, see Appendix D. However the system is unstable which tends to prove that firms’ prices may never reach this case. This affine pricing does not exist either when using the KM constraint, see Appendix C.

We analyzed all interior solutions, and all situations of maximal investment with affine prices. Only three solutions coexist:

- The shadow price of the capital \( v \) is exactly \( \frac{1}{\beta} \) and the average capital of firms is determined by \( R \) through equation (21). Since \( K = E + D = E + v \gamma E \), we can deduce all equilibrium values.
- The shadow price of the capital \( v \) strictly exceed \( \frac{1}{\beta} \), the firms invest as much as possible. The value of \( v \) is given by equation (51) and the value of \( R \) by equation (50).
- The parameters satisfy \( \pi = \frac{\delta \gamma}{1 - \beta + \delta \gamma} \) and the value of the equity is a pure bubble, the real capital is only composed of debt.

Since all these equilibrium values are stable, there is no reason to choose one or the others. From a welfare point of view, the first case has lower values of \( R \) and therefore a higher capital. From an investor point of view, the bubble allows to generate income with a zero equity. The solution may depend on the endowments at the initial time of the problem.
5.1 Stochastic equilibrium

This section is the beginning of an “upgrade” of the 5th chapter of the PhD dissertation. The objective is to study how prices may switch from an equilibrium to another one, to derive the new equilibrium prices, and to study the path of the whole economy (prices, capital, debts) when the equilibrium changes. This first results are encouraging, but this part is not finished yet.

We suppose that the whole economy is located at the linear pricing and the Tobin’s Q is larger than $\frac{1}{\beta}$. We want to introduce a possibility of switching from an equilibrium to another one. Like Miao and Wang (2011) or Kocherlakota (2009), we suppose that there exists an exogenous probability $\theta$ such that the economy remains in the linear pricing at the following period, and $(1 - \theta)$ that the economy “crashes” on the case where the Tobin’s Q is $\frac{1}{\beta}$ (the case when the investment constraint is not binding). We also suppose that the crash is irreversible. We want to determine the new pricing of the economy in this context. The new Bellman equation can be stated as follows. Let $W_t(E^m_t)$ be the price of the equity $E^m_t$ of firm $m$ at time $t$:

\[
W_t(E^m_t) = \max_{I^m_t} R_tE^m_t + (R_t - r_t)D^m_t - \pi I^m_t \\
+ \pi \gamma W_{t+1} \left( E^m_t (1 - \delta) - \delta D^m_t + I^m_t \right) + (1 - \pi) \beta \theta W_{t+1} \left( E^m_t (1 - \delta) - \delta D^m_t + I^m_t \right) \\
+ \pi \left( 1 - \pi \right) \beta \left( E^m_t (1 - \delta) - \delta D^m_t + I^m_t \right) + (1 - \pi) \beta \left( 1 - \theta \right) \frac{1}{\beta} \left( E^m_t (1 - \delta) - \delta D^m_t \right);
\]

where $\frac{1}{\beta}$ comes from the interior solution in the previous section. The debt is secured by the value of a part of the equity. Let us suppose that the constraint is type KM: $D^m_t \leq \gamma \text{Value}(E^m_t)$. The upper limit of the investment also depends of the probability of crash:

\[
I^m_t \leq R_tE^m_t + (R_t - r_t) \left( \theta \gamma W_t(E^m_t) + (1 - \theta) \frac{\gamma}{\beta} E^m_t \right). \tag{57}
\]

We suppose that the new pricing $W_t(E^m_t)$ can be written as an affine pricing:

\[
W_t(E^m_t) = w_t E^m_t + z_t. \tag{56}
\]

This pricing would only exist if the shadow price $w_t$ of the equity is larger than $\frac{1}{\beta}$. As a consequence the investment constraint
is supposed to be binding. The Bellman equation (56) becomes:

\[ w_t E_t^m + z_t = \]

\[ (1 - \pi) \left( R_t E_t^m + (R_t - r_t) \gamma \left( \theta(w_t E_t^m + z_t) + \frac{1 - \theta}{\beta} E_t^m \right) \right) \]

\[ + \pi \beta \theta w_{t+1} \left( (1 - \delta) E_t^m - \delta \gamma \left( \theta(w_t E_t^m + z_t) + \frac{1 - \theta}{\beta} E_t^m \right) \right) \]

\[ + R_t E_t^m + (R_t - r_t) \gamma \left( \theta(w_t E_t^m + z_t) + \frac{1 - \theta}{\beta} E_t^m \right) + \pi \beta \theta z_{t+1} \]

\[ + (1 - \pi) \beta \theta w_{t+1} \left( (1 - \delta) E_t^m - \delta \gamma \left( \theta(w_t E_t^m + z_t) + \frac{1 - \theta}{\beta} E_t^m \right) \right) \]

\[ + (1 - \pi) \beta \theta z_{t+1} \]

\[ + \pi (1 - \theta) \left( (1 - \delta) E_t^m - \delta \gamma \left( \theta(w_t E_t^m + z_t) + \frac{1 - \theta}{\beta} E_t^m \right) \right) \]

\[ + R_t E_t^m + (R_t - r_t) \gamma \left( \theta(w_t E_t^m + z_t) + \frac{1 - \theta}{\beta} E_t^m \right) \]

\[ + (1 - \pi)(1 - \theta) \left( (1 - \delta) E_t^m - \delta \gamma \left( \theta(w_t E_t^m + z_t) + \frac{1 - \theta}{\beta} E_t^m \right) \right). \]

Using the same method as before, we separate the terms depending on \( E_t^m \) from the others, and we get two difference equations on the shadow price \( w_t \) and the shift term \( z_t \):

\[ w_t (1 - (R_t - r_t) \gamma \theta(1 - \pi \theta) + \delta \gamma \theta(1 - \theta)) \]

\[ + w_t w_{t+1} \beta \theta (\delta \gamma \theta - \pi (R_t - r_t) \gamma \theta) \]

\[ = w_{t+1} \beta \theta \left( 1 - \delta - \delta \gamma \frac{1 - \theta}{\beta} + \pi (R_t + (R_t - r_t) \gamma \frac{1 - \theta}{\beta}) \right) \]

\[ + \left( (R_t - r_t) \gamma \frac{1 - \theta}{\beta} \right) (1 - \pi \theta) + R_t (1 - \theta \pi) + \left( 1 - \delta - \delta \gamma \frac{1 - \theta}{\beta} \right) (1 - \theta); \]

\[ \quad \text{(59)} \]

and

\[ z_t (1 - (R_t - r_t) \gamma \theta(1 - \theta \pi + \theta \pi \beta w_{t+1}) + \delta \gamma \theta(\beta \theta w_{t+1} + 1 - \theta)) = \beta \theta z_{t+1}. \]

\[ \quad \text{(60)} \]

These two equations corresponds to the equations of the maximal investment
when $\theta = 1$. Both $w_t$ and $z_t$ must satisfy the transversality conditions:

\begin{align*}
    \bullet & \beta^t w_t E_t \to_{t \to \infty} 0; \\
    \bullet & \beta^t z_t \to_{t \to \infty} 0. 
\end{align*}

Using equation (60), we deduce that the transversality condition imposes:

\[ \frac{1}{\theta} (1 - (R_t - r_t) \gamma \theta (1 - \theta \pi + \theta \pi \beta w_{t+1}) + \delta \gamma \theta (\beta \theta w_{t+1} + 1 - \theta)) \leq 1. \]  

(62)

This leads to:

\[ (R_t - r_t)(1 - \theta \pi + \theta \pi \beta w_{t+1}) - \delta (1 - \theta + \beta \theta w_{t+1}) \geq \frac{1 - \theta}{\gamma \theta}. \]  

(63)

As we can guess, the condition is more likely to be satisfied when $\theta \to 1$. On the contrary, when there is a high probability of crash ($\theta \to 0$), the affine component cannot satisfy the transversality condition, and therefore the problem imposes $z = 0$.

The last equation of the firms’ problem comes from the aggregation of the equity:

\[ E_{t+1} = (1 - \delta) E_t - \delta D_t^n + \pi (R_t E_t + (R_t - r_t) \gamma W_t(E_t)). \]  

(64)

Using the value of the debt and the price, the equation becomes:

\[ E_{t+1} = E_t \left(1 - \delta + \pi R_t + \gamma \left(\theta w_t + \frac{1 - \theta}{\beta}\right) (\pi (R_t - r_t) - \delta)\right) + \gamma \theta z_t (\pi (R_t - r_t) - \delta). \]  

(65)

We can look for a steady state of these three difference equations.

5.2 Affine Price

Looking at the equation on the average equity (65), we can deduce some information on the value of $R$. If there is a stationary equilibrium such that $z \neq 0$, we can distinguish three cases:

\begin{itemize}
    \item $\pi (R - r_\infty) - \delta = 0$ and then the multiplier term before $E_t$ is equal to 1. This leads to a contradiction because it would also impose $1 - \delta + \pi R = 1$, which gives $r_\infty = 0$.
    \item $\pi (R - r_\infty) - \delta > 0$. To keep constant the equilibrium value of the equity,
this imposes: \((1 - \delta + \pi R_t + \gamma \left( \theta w_t + \frac{1-\theta}{\beta} \right) (\pi (R_t - r_t) - \delta) \) < 1. This is actually impossible because it is equivalent to: 
\[-\delta + \pi R + \gamma (\theta w + \frac{1-\theta}{\beta}) (\pi (R - r) - \delta) < 0 \Rightarrow \pi R < \delta.\]

- We deduce that \(\pi (R - r_{\infty}) - \delta < 0\) and \((1 - \delta + \pi R_t + \gamma \left( \theta w_t + \frac{1-\theta}{\beta} \right) (\pi (R_t - r_t) - \delta) \) > 1. This gives a small interval for \(\pi R\):

\[
\delta + \pi r_{\infty} \frac{\gamma (\theta w + \frac{1-\theta}{\beta})}{1 + \gamma (\theta w + \frac{1-\theta}{\beta})} < \pi R < \delta + \pi r_{\infty}. \tag{66}
\]

Using the (60) gives the equilibrium value of \(w\) as a function of \(R\):

\[
w = \frac{1 - \beta \theta + \delta \gamma \theta (1 - \theta) - (R - r) \gamma \theta (1 - \theta \pi)}{\beta \theta^2 \gamma (\pi (R - r) - \delta)}. \tag{67}
\]

Incoming: stability of the equilibria and numerical simulations.

5.3 Linear Price

We can also suppose that the shift term of the price \(z_t\) is zero. It only remains the equation on the equity (65) and the equation on the shadow price \(w_t\). We look for the equilibrium values of the model. If the equity is constant, we have:

\[-\delta + \pi R + \gamma \left( \theta w + \frac{1-\theta}{\beta} \right) (\pi (R - r_{\infty}) - \delta) = 0. \tag{68}
\]

6 Conclusion

In a production economy, where the capital of firms is composed of equity and debt, limitation of the collateral and stochastic investment opportunities bring firms to invest a lot when possible. There exist two ways of pricing firms. One of them corresponds to a higher average level of capital, where firms’ prices correspond to the Tobins’ Q. In this case, the level of equity is high enough, such that firms are not penalized by the stochastic investment. The other equilibrium corresponds to a lower average capital, and maximal investment of firms. In this equilibrium, firms’ prices are higher, and help firms to reach the optimal amount of investment. Furthermore, these prices are very sensitive to the interest rates on loans. Positive interest-rate shocks
decrease the equity, but increase the price, allowing to keeps a high level of debt. When interest rate shocks are persistent, firms’ prices may be discontinuous, and somehow illustrate market’s crashes. Even if price of firms exceed their usual Tobin’s Q value, there are no bubbles in prices as soon as the equity is positive. Nevertheless, an unusual situation may happen depending on the investments parameters: the equity does not exist, but has a net positive price, this is a bubble. The bubble allows for borrowing, and with the debt used as capital, firms can generate strictly positive cash flows. In this model of long-term debts in the capital of firms, we also proved that the choice of the borrowing constraint does not influence the equilibrium prices. In addition, affine prices of firms do not exist either.

There are many ways to deepen the analysis of this model. First, it would be relevant to get closer to the bubble, even if asymptotic properties (zero equity, infinite shadow price, continuous price) represent a technical challenge. Then, it would be interesting to investigate a global equilibrium in which the financial intermediaries collect the households deposits’ and lend them to the firms. This would endogenize the supply of loans, and the interest rate. By limiting the supply of loans, pricings of firms could be different. Finally, to submit this long-term debts model to a review, we could recreate a two-state problem, by introducing a probability of falling from the linear-price to the Q-price.
A Boundary solution with the KM constraint

We apply the same reasoning as in section 5.2.3, starting from the Bellman equation at the end of section 2, equation (17). The equation on $v_t$ does not change, but the one on $b_t$ becomes:

$$b_t [1 - \gamma((R_t - r_t)(1 - \pi) + ((R_t - r_t)\pi - \delta)\beta v_{t+1})] = \beta b_{t+1}. \quad (69)$$

We remark that the domain of transversality is the same as the one resulting from the MW constraint. The equation on the average equity as well as the equation on the rental rate of the capital evolve:

$$E_{t+1} = (1 - \delta) E_t - \delta \gamma (v_t E_t + b_t) + \pi (R_t E_t + (R_t - r_t)\gamma (v_t E_t + b_t)), \quad (70)$$

$$R_t = \alpha K_t^{\alpha-1} = \alpha (E_t (1 + \gamma v_t) + \gamma b_t)^{\alpha-1}. \quad (71)$$

The steady state value change because the equation on the bubble leads to:

$$v = \frac{1 - \beta - \gamma (R - r_\infty)(1 - \pi)}{(\pi (R - r_\infty) - \delta)\beta \gamma}. \quad (72)$$

The equation (76) does not change. The new value of $R$ is given by:

$$R = \frac{\delta (1 - \beta) + \delta \gamma r_\infty (1 - \pi)}{\pi (1 - \beta)}. \quad (73)$$

The link between $A$ and $b$ also changes by a factor $\gamma$:

$$b = \frac{E^-\delta - \delta v \gamma + \pi R + \pi (R - r_\infty)\gamma v}{\gamma (\delta - \pi (R - r_\infty)). \quad (74)$$

We take the same parameters values as before and draw the values of our valuation components. $R$ behaves the same way as the one with the MW constraint. There are some problems for $v$ because $v$ remains negative for any value of $R$. This means that $v < \frac{1}{\beta}$ and the KM constraint rules out the non-linear pricing, as in the previous chapter.

B Affine pricing with Miao and Wang constraint

As proved in chapter 4, following Miao and Wang (2011), there exists steady state of the model in which $b \neq 0$. Any firm has a positive $b$ shift in the
price, which creates a distortion, especially for firms with low capital. This originality of pricing comes from the use of the MW constraint. We check in our long-term debts model whether the use of this constraint may create such a non-linear pricing.

We compute the steady state values when the bubble component is not zero. The equation \((44)\) gives the value of \(v\):

\[
v = \frac{1 - \beta - (R - r_\infty)(1 - \pi)}{\beta((R - r_\infty)\pi - \delta)}.
\]

From its own equation, we also have a value of \(v\):

\[
\beta \gamma v^2(\delta - \pi(R - r_\infty)) + v - v\gamma(R - r_\infty)(1 - \pi) - v\beta \pi R - \beta v(1 - \delta) - R(1 - \pi) = 0.
\]

These two expressions of \(v\) lead to the value of \(R\):

\[
R = \frac{(1 - \gamma(1 - \beta) - \beta(1 - \delta))(1 - \beta + r_\infty(1 - \pi))}{\beta \pi(1 - \beta) + (1 - \pi)(1 - \beta \delta - \gamma(1 - \beta) - \beta(1 - \delta))}.
\]

The value of \(v\) is given by the value of \(R\) and the link between \(b\) and \(E\) comes from equation \((46)\):

\[
b = E \frac{-\delta - \delta v\gamma + \pi R + \pi(R - r_\infty)\gamma v}{\delta - \pi(R - r_\infty)}.
\]

The equation \((48)\) together with \((78)\) determines \(A\) and \(b\). We represent the main values of \(b\) and \(v\). We take \(\alpha = 0.4\), \(\delta = 0.025\), \(\beta = 0.96\) and \(r = 4\%\). On the values of \(v\), we limit to an arbitrary value \((20)\) because it also diverges like in the linear pricing. The dark area is forbidden because \(v \leq \frac{1}{\beta}\).

The shadow price of the non-linear pricing does not exist for the same values as the shadow price of the linear pricing: low values of the investment probabilities are privileged. This corresponds to the results of the model of Miao and Wang (2011).

We represent the bubble component on Figure 10, again the dark value corresponds to the negative values of \(b\). Surprisingly the bubble is increasing with respect to \(\pi\) which contrasts with the previous chapter. This may correspond to acceleration phenomena, such as new technologies, for example the dot-com bubble.
We must also check that the value of the rental rate of the capital verifies \( R > r_\infty = 4\% \): this condition is satisfied while \( \gamma > 25\% \) or \( \pi < 75\% \), on Figure 11.

The last point is the average value of the equity \( E \) that needs to remain above 0 on Figure 12:

We intersect all the previous domains to get the complete zone on Figure 13 where the non-linear pricing exists in the economy.

Everywhere this steady state exists, it is unstable: the system of the 3 linearized equations has 3 eigenvalues larger than 1 in modulus. The prices can never reach this equilibrium. For all values of the parameters \( \pi \) and \( \gamma \) such that the non-linear pricing exists, the equation (44) on the shift term \( b \) verifies the transversality condition if \( b \neq 0 \):

\[
(R_t - r_t)(1 - \pi + \pi \beta v_{t+1}) - \delta \beta v_{t+1} > 0.
\]  

(79)

This shows that there is no equilibrium affine pricing using Miao and Wang constraint.

When the capital of firms is composed of equity and debt, the choice of the borrowing constraint does not matter, the prices of firms are the same, and there exists only a unique linear pricing when stochastic investment opportunities force the firms to invest “as much as possible”.
Figure 10: Affine component of the valuation $b$, non-linear pricing, with respect to $\pi$ and $\gamma$
Figure 11: Rental rate of the capital $R$, non-linear pricing, with respect to $\pi$ and $\gamma$. 
Figure 12: Average equity $E$, non-linear pricing, with respect to $\pi$ and $\gamma$
Figure 13: Existence domain of the non-linear pricing, with respect to $\pi$ and $\gamma$
References


